Amplified Monte Carlo simulation methods for analysis of systems reliability

A. Naess, NTNU
B.J. Leira, NTNU

Prague-May-2011

NTNU

Introduction

- A Monte Carlo based method for estimating reliability which aims at reducing the computational cost is proposed.
- The method exploits the regularity of tail probabilities to set up an approximation procedure for the prediction of the far tail failure probabilities. It is based on estimating the failure probabilities by Monte Carlo simulation at more moderate failure probability levels.

The usefulness and accuracy of the estimation method will be illustrated by application to specific examples of systems with a large number of limit states.

NTNU

A motivating example

- Let R and S be two independent Gaussian variables representing the capacity and the demand, respectively, in the simplest safety format.
- That is, the safety margin $M = R - S$. Failure is assumed to occur when $M \leq 0$.
- The probability of failure is then $p_f = \text{Prob}(M \leq 0)$.
- In the present case, $p_f = \Phi(-\beta)$, where $\Phi(\cdot)$ denotes the cumulative probability distribution of an $N(0, 1)$ variable and β denotes the Cornell safety index.
- That is, $\beta = \mu_M/\sigma_M$, where $\text{E}[M] = \mu_M$ and $\text{Var}[M] = \sigma_M^2$. $\mu_M = \mu_R - \mu_S$ and $\sigma_M = \sqrt{\sigma_R^2 + \sigma_S^2}$.

A motivating example

- The safety margin M will now be extended to a parametrized class of safety margins in the following way

$$M(\lambda) = M - \mu_M(1-\lambda)\,,$$

where the scaling parameter λ satisfies $0 \leq \lambda \leq 1$.

- It follows that $M = M(1)$, and the Cornell index $\beta(\lambda)$ of $M(\lambda)$ is seen to be given as $\beta(\lambda) = \lambda\beta$ since $\mathrm{E}[M(\lambda)] = \lambda\mu_M$ and $\mathrm{Var}[M(\lambda)] = \sigma_M^2$.

- Hence, the failure probability $p_f(\lambda) = \mathrm{Prob}(M(\lambda) \leq 0) = \Phi(-\lambda\beta)$.

A motivating example

- Assuming that $p_f = p_f(1)$ is small, e.g. less than about 10^{-3}, it is obtained that:

$$p_f(\lambda) = \Phi(-\lambda\beta) \underset{\lambda\to 1}{\approx} \left(\frac{1}{\lambda\beta} - \frac{1}{(\lambda\beta)^3} + \frac{3}{(\lambda\beta)^5}\right)\phi(\lambda\beta)$$
$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\lambda\beta} - \frac{1}{(\lambda\beta)^3} + \frac{3}{(\lambda\beta)^5}\right)\exp\left\{-\frac{\beta^2\lambda^2}{2}\right\},$$

  where ϕ is the probability density function of an $N(0,1)$ variable.

- In fact, a uniformly close approximation (with an error less than $7.5 \cdot 10^{-8}$) which is similar to the right hand side of this equation can be given for all positive values of the argument z of $\Phi(-z)$.

- For any safety margin for which a FORM or SORM approximation applies after transformation to normalized Gaussian space, it is realized that the failure probability $p_f(\lambda)$ will be given by an equation somewhat similar to the equation above.

The general case

- In the general case the safety margin $M = G(X_1, \ldots, X_n)$ is expressed in terms of n basic variables.
- A similar analysis as for the simple example cannot be easily done without making some assumptions.
- However, motivated by the simple example and the ensuing comment, we shall make the following assumption about the behaviour of the failure probability,

$$p_f(\lambda) \underset{\lambda \to 1}{\approx} q(\lambda) \exp\left\{ -a(\lambda - b)^c \right\},$$

where the function $q(\lambda)$ is slowly varying compared with the exponential function $\exp\{-a(\lambda - b)^c\}$.

NTNU

The general case

- The practical importance of this relation, if it applies, is that the target failure probability $p_f = p_f(1)$ can be obtained from values of $p_f(\lambda)$ for $\lambda < 1$.
- Our focus in this presentation is on methods for estimating p_f by Monte Carlo simulation.
- The observation above may then be significant, as it may be easier to estimate the failure probabilities $p_f(\lambda)$ for $\lambda < 1$ accurately than the target value since they are larger and hence require less simulations.
- Fitting the parametric form for $p_f(\lambda)$ to the estimated values would then allow us to provide an estimate of the target value by extrapolation.

System reliability

- The modified series system reliability expressed in terms of the failure probability can then be written as,

$$p_f(\lambda) = \text{Prob}\Big(\bigcup_{j=1}^{m}\{M_j(\lambda) \leq 0\}\Big)\,,$$

- The failure probability for the parallel system,

$$p_f(\lambda) = \text{Prob}\Big(\bigcap_{j=1}^{m}\{M_j(\lambda) \leq 0\}\Big)\,.$$

- Any system can be written as a series system of parallel subsystems. Then $(k(1) + \ldots + k(l) = m)$

$$p_f(\lambda) = \text{Prob}\Big(\bigcup_{j=1}^{l}\bigcap_{i=1}^{k(j)}\{M_{ji}(\lambda) \leq 0\}\Big)\,,$$

Monte Carlo based reliability estimation

- It is now tentatively proposed to replace $q(\lambda)$ by a suitable constant value, q say, for tail values of λ.
- Hence, we will investigate the viability of the following approximation,

$$p_f(\lambda) \approx q \exp\{-a(\lambda - b)^c\}, \quad \text{for } \lambda_0 \leq \lambda \leq 1,$$

for a suitable choice of λ_0.

Monte Carlo based reliability estimation

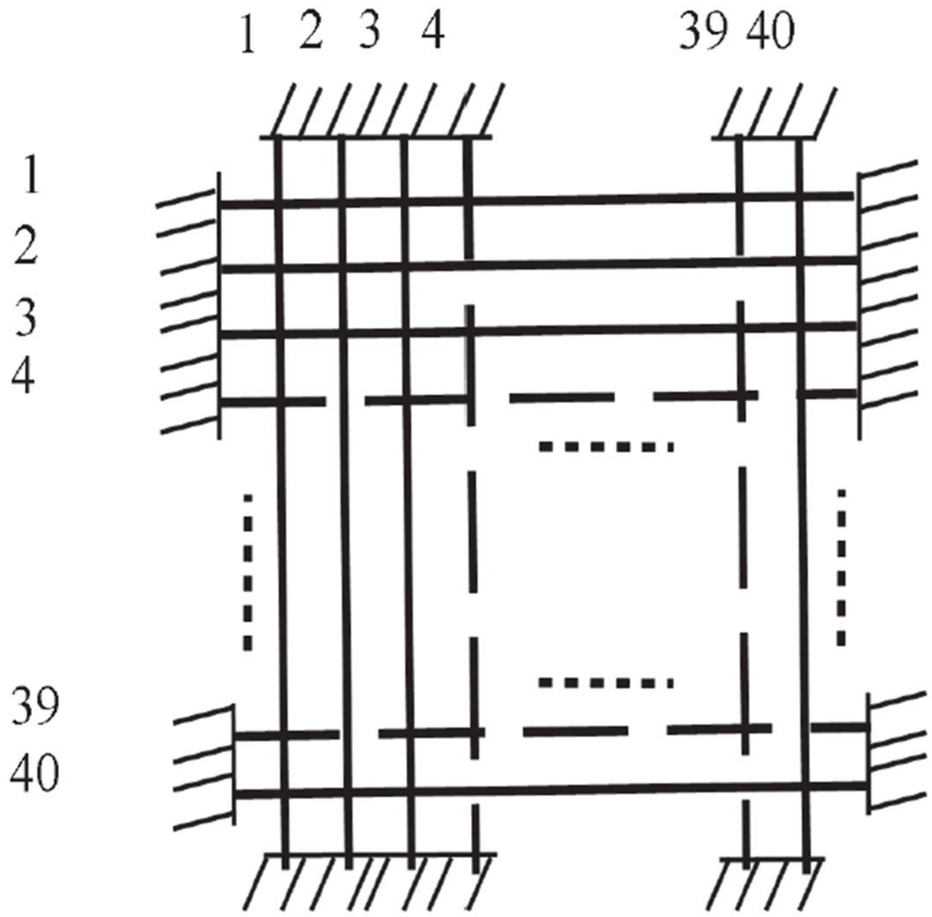
- The optimal values of the parameters q, a, b, c is found by minimizing the following mean square error function,

$$F(q,a,b,c) = \sum_{j=1}^{M} w_j \left(\log \hat{p}_f(\lambda_j) - \log q + a(\lambda_j - b)^c \right)^2 ,$$

where $\lambda_0 \le \lambda_1 < \ldots < \lambda_M < 1$ denotes the set of λ values where the failure probability is empirically estimated.

- $w_j, j = 1, \ldots, M$, denote weight factors that put more emphasis on the more reliable data points, alleviating the heteroscedasticity of the estimation problem at hand.
- We use $w_j = \left(\log C^+(\lambda_j) - \log C^-(\lambda_j) \right)^{-\theta}$ with $\theta = 1$ or 2, combined with a Levenberg-Marquardt least squares optimization method.

Example: 40x40 beam girder system (grillage)



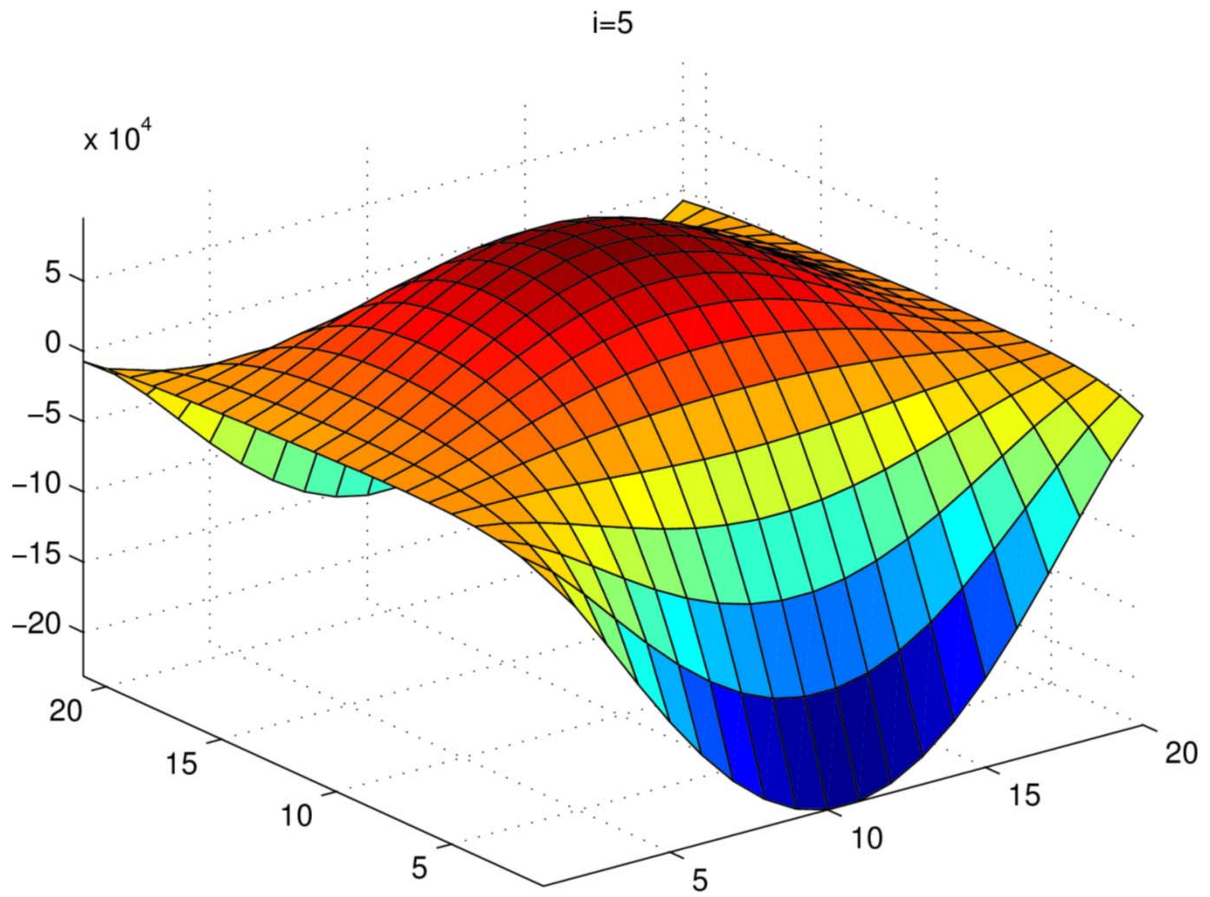
NTNU

Example: 40x40 beam girder system (grillage)

- The grillage structure is modelled as a series system with 6540 limit state functions and a total of 4880 random variables. Each beam element has a rectangular cross section of dimension 0.05 m times 0.12 m and a length of 0.5 m.
- All vertical load components are normally distributed with mean 420N and standard deviation 126N. They are pairwise equicorrelated. The yield stress is modelled as lognormally distributed random variables, mean = 380 MPa, standard deviation = 19 MPa
- The limit state function for each node of the grillage has been formulated as

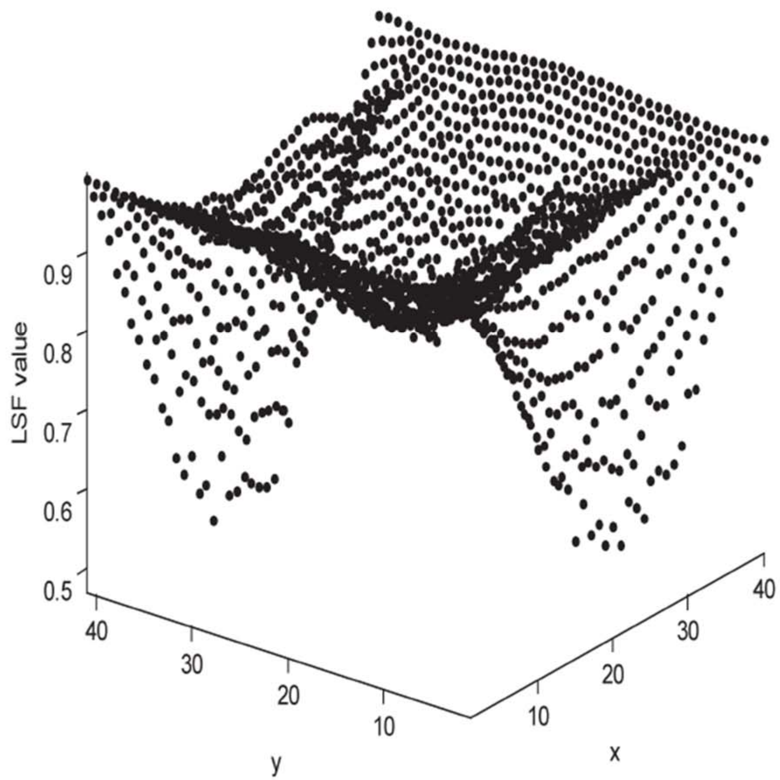
$$g(\mathbf{X}) = 1 - \left\{ \left|\frac{\sigma_B}{\sigma_{B,cr}}\right| + \left|\frac{\tau_V}{\tau_{V,cr}}\right| + \left|\frac{\tau_T}{\tau_{T,cr}}\right| \right\}$$

Example: 40x40 beam girder system (grillage)
Bending moment distribution



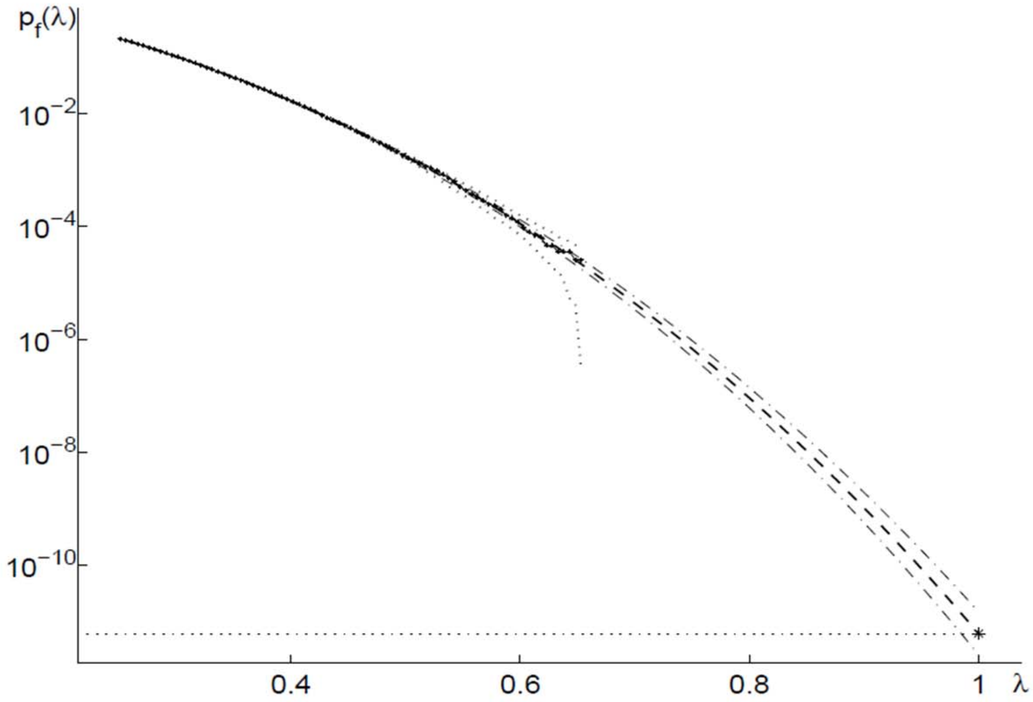
Example: 40x40 beam girder system (grillage)

Limit state function values for Node 2 of transverse beam elements.



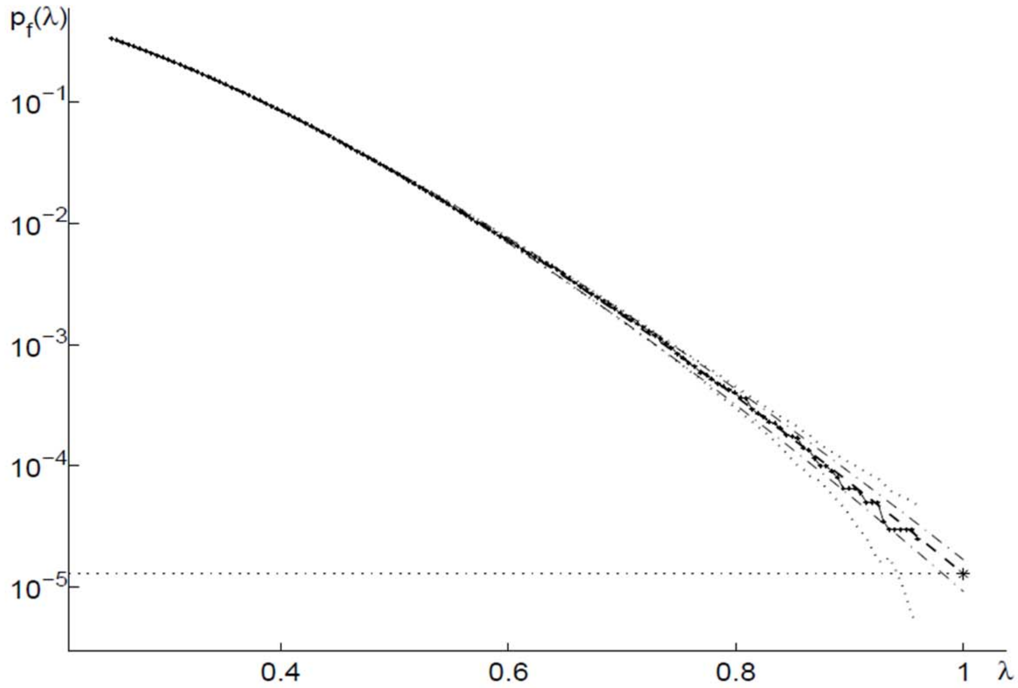
Example: 40x40 beam girder system (grillage)

Plot of the probability of failure p_f versus λ for load correlation length 1.5m: Monte Carlo (●); fitted optimal curve (– –); reanchored empirical confidence band (· · ·); fitted confidence band (– ·).

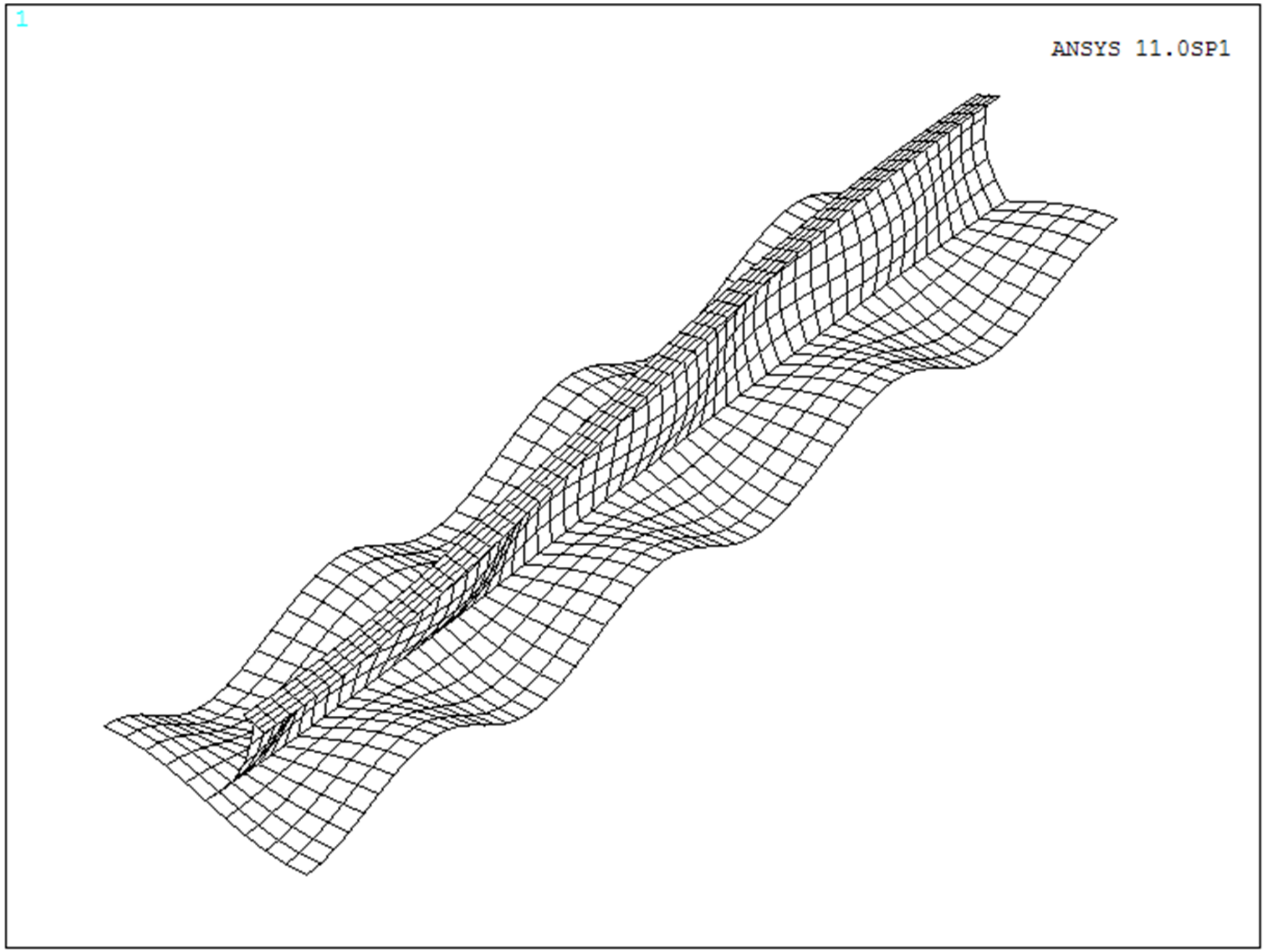


Example: 40x40 beam girder system (grillage)

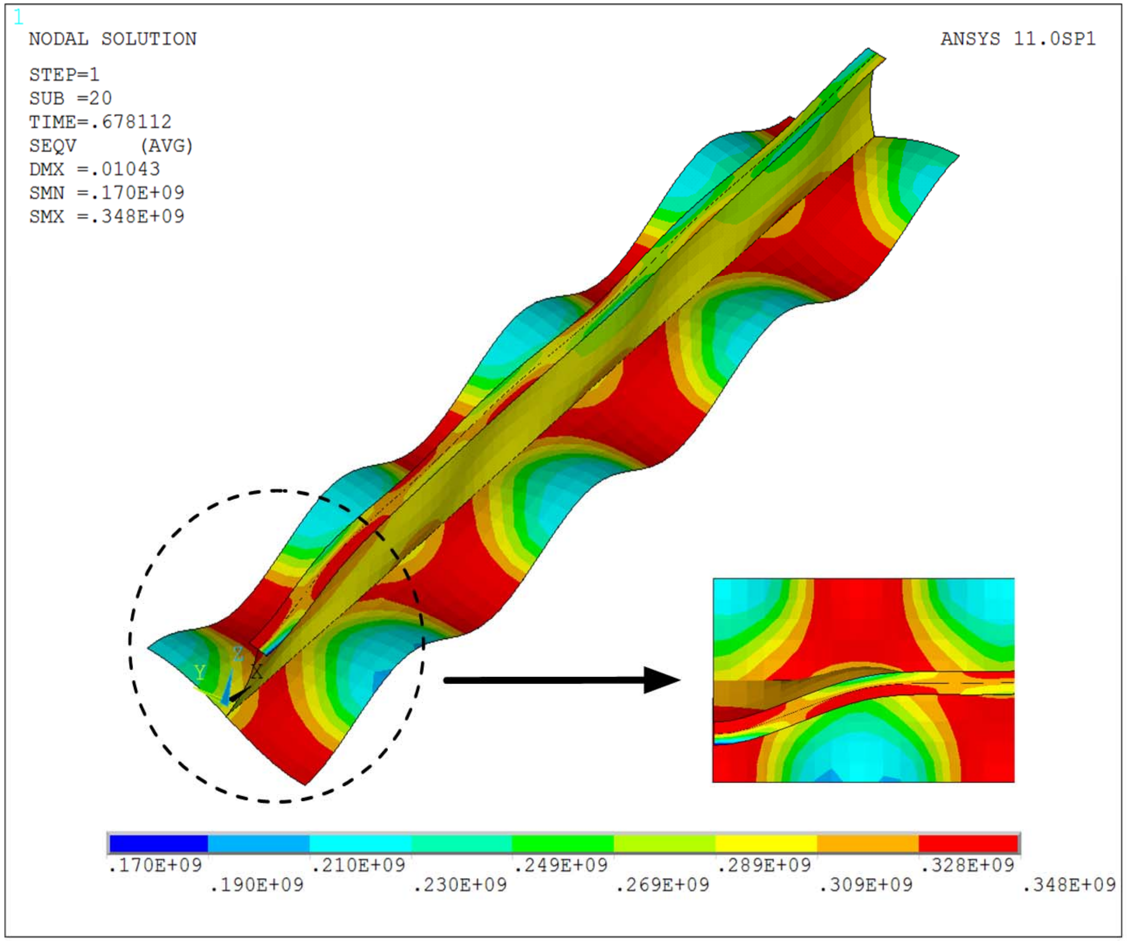
Plot of the probability of failure p_f versus λ for load correlation length 2.5m: Monte Carlo (●); fitted optimal curve (– –); reanchored empirical confidence band (· · ·); fitted confidence band (– ·).



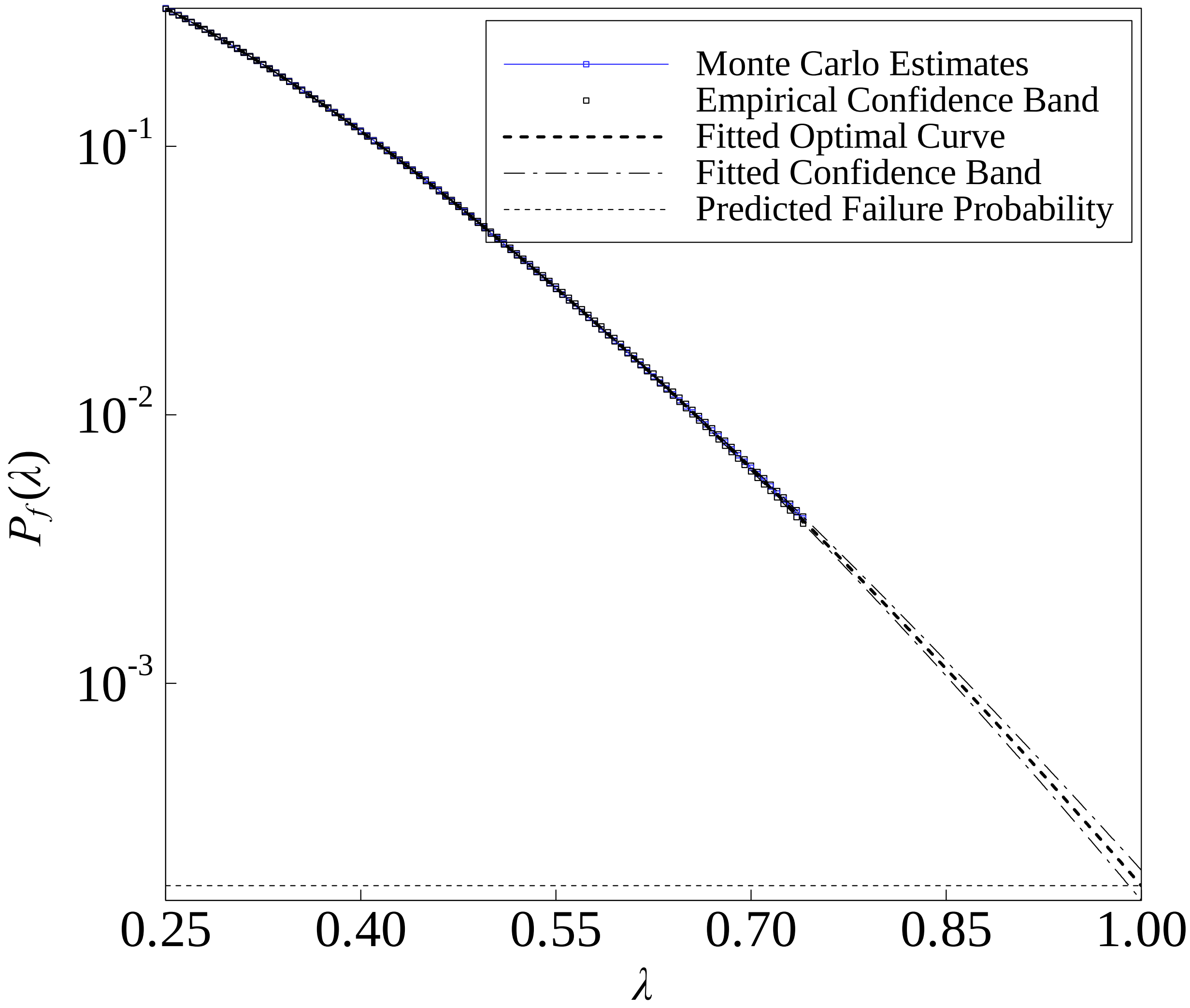
Example 2: Buckling of stiffened panel by non-linear FEM



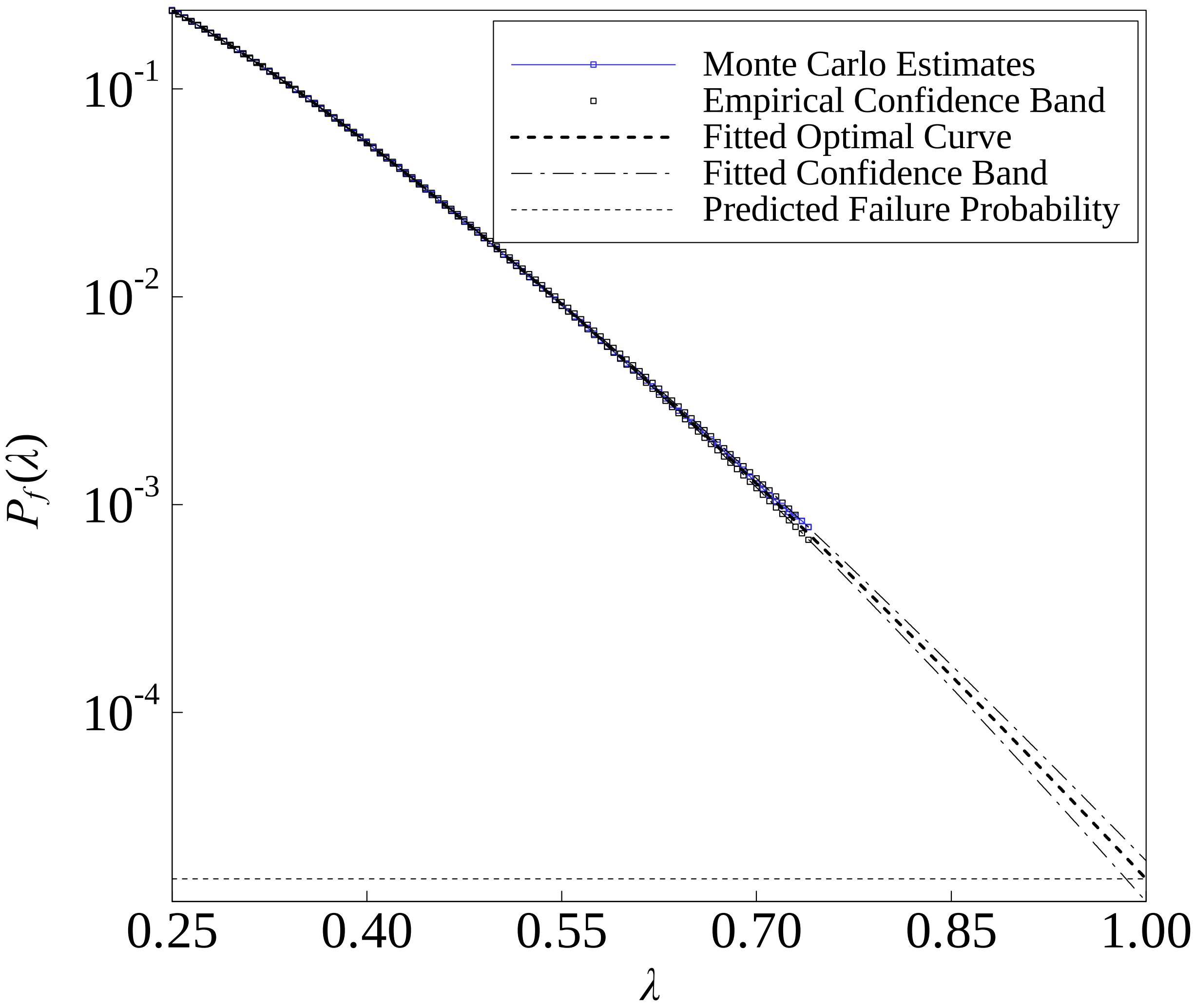
Example 2:
Buckling of stiffened panel by non-linear FEM and response surface methods



Example 2: Corroded scantlings



Example 2: Intact scantlings



NTNU

Example 2: Crude Monte Carlo

	Component Analysis		*System Analysis*	
	Corroded	*Intact*	*Corroded*	*Intact*
N	10^9	10^9	10^8	10^8
P_f	$4.33 \cdot 10^{-5}$	$4.07 \cdot 10^{-6}$	$2.36 \cdot 10^{-4}$	$2.40 \cdot 10^{-5}$
β	3.93	4.46	3.50	4.07
ΔP_f*	±0.9%	±3.1%	±1.3%	±4.0%

*Accuracy of the failure probability estimate with 95% confidence.

NTNU

Conclusions

- From the examples studied, one can (tentatively) conclude that the proposed Monte Carlo based method for reliability calculations appears to be accurate (enough) and robust, while it is simple and practical to use.